Mathematics
Higher level
Paper 3 – discrete mathematics

Friday 18 November 2016 (morning)

1 hour

Instructions to candidates

- Do not open this examination paper until instructed to do so.
- Answer all the questions.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- A graphic display calculator is required for this paper.
- A clean copy of the **mathematics HL and further mathematics HL formula booklet** is required for this paper.
- The maximum mark for this examination paper is **[60 marks]**.

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Please start each question on a new page. Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. In particular, solutions found from a graphic display calculator should be supported by suitable working. For example, if graphs are used to find a solution, you should sketch these as part of your answer. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

1. [Maximum mark: 17]

In this question the notation $(a_n a_{n-1} \ldots a_2 a_1 a_0)_b$ is used to represent a number in base b, that has unit digit of a_0. For example $(2234)_5$ represents $2 \times 5^3 + 2 \times 5^2 + 3 \times 5 + 4 = 319$ and it has a unit digit of 4.

(a) Let x be the cube root of the base 7 number $(503231)_7$.

(i) By converting the base 7 number to base 10, find the value of x, in base 10.

(ii) Express x as a base 5 number. [7]

(b) Let y be the base 9 number $(a_n a_{n-1} \ldots a_1 a_0)_9$. Show that y is exactly divisible by 8 if and only if the sum of its digits, $\sum_{i=0}^{n} a_i$, is also exactly divisible by 8. [7]

(c) Using the method from part (b), find the unit digit when the base 9 number $(321321321)_9$ is written as a base 8 number. [3]

2. [Maximum mark: 8]

In this question no graphs are required to be drawn. Use the handshaking lemma and other results about graphs to explain why,

(a) a graph cannot exist with a degree sequence of $1, 2, 3, 4, 5, 6, 7, 8, 9$; [2]

(b) a simple, connected, planar graph cannot exist with a degree sequence of $4, 4, 4, 4, 5, 5$; [3]

(c) a tree cannot exist with a degree sequence of $1, 1, 2, 2, 3, 3$. [3]

3. [Maximum mark: 16]

In a computer game, Fibi, a magic dragon, is climbing a very large staircase. The steps are labelled $0, 1, 2, 3 \ldots$.
She starts on step 0. If Fibi is on a particular step then she can either jump up one step or fly up two steps. Let u_n represent the number of different ways that Fibi can get to step n. When counting the number of different ways, the order of Fibi's moves matters, for example jump, fly, jump is considered different to jump, jump, fly. Let $u_0 = 1$.

(a) Find the values of u_1, u_2, u_3. [3]

(b) Show that $u_{n+2} = u_{n+1} + u_n$. [2]

(c) (i) Write down the auxiliary equation for this recurrence relation.

(ii) Hence find the solution to this recurrence relation, giving your answer in the form $u_n = A\alpha^n + B\beta^n$ where α and β are to be determined exactly in surd form and $\alpha > \beta$. The constants A and B do not have to be found at this stage. [5]

(d) (i) Given that $A = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)$, use the value of u_0 to determine B.

(ii) Hence find the explicit formula for u_n. [3]

(e) Find the value of u_{20}. [1]

(f) Find the smallest value of n for which $u_n > 100\,000$. [2]

4. [Maximum mark: 19]

The simple, complete graph $\kappa_n (n > 2)$ has vertices $A_1, A_2, A_3, \ldots, A_n$. The weight of the edge from A_i to A_j is given by the number $i + j$.

(a) (i) Draw the graph κ_4 including the weights of all the edges.

(ii) Use the nearest-neighbour algorithm, starting at vertex A_1, to find a Hamiltonian cycle.

(iii) Hence, find an upper bound to the travelling salesman problem for this weighted graph. [4]

(b) Consider the graph κ_5. Use the deleted vertex algorithm, with A_5 as the deleted vertex, to find a lower bound to the travelling salesman problem for this weighted graph. [5]

Consider the general graph κ_n.

(c) (i) Use the nearest-neighbour algorithm, starting at vertex A_1, to find a Hamiltonian cycle.

(ii) Hence find and simplify an expression in n, for an upper bound to the travelling salesman problem for this weighted graph. [7]

(d) By splitting the weight of the edge A_iA_j into two parts or otherwise, show that all Hamiltonian cycles of κ_n have the same total weight, equal to the answer found in (c)(ii). [3]
